5-6 The Quadratic Formula and the Discriminant

Main Ideas

- Solve quadratic equations by using the Quadratic Formula.
- Use the discriminant to determine the number and type of roots of a quadratic equation.

New Vocabulary

Quadratic Formula
discriminant

GET READY for the Lesson

Competitors in the 10-meter platform diving competition jump upward and outward before diving into the pool below. The height h of a diver in meters above the pool after t seconds can be approximated by the equation $h = -4.9t^2 + 3t + 10$.

Quadratic Formula You have seen that exact solutions to some quadratic equations can be found by graphing, by factoring, or by using the Square Root Property. While completing the square can be used to solve any quadratic equation, the process can be tedious if the equation contains fractions or decimals. Fortunately, a formula exists that can be used to solve any quadratic equation of the form $ax^2 + bx + c = 0$. This formula can be derived by solving the general form of a quadratic equation.

$$ax^2 + bx + c = 0 \quad \text{General quadratic equation}$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \text{Divide each side by } a.$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \quad \text{Subtract } \frac{c}{a} \text{ from each side.}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} \quad \text{Complete the square.}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad \text{Factor the left side. Simplify the right side.}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{Square Root Property}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{Subtract } \frac{b}{2a} \text{ from each side.}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{Simplify.}$$

This equation is known as the **Quadratic Formula**.

Reading Math

Quadratic Formula The Quadratic Formula is read *x equals the opposite of b, plus or minus the square root of b squared minus 4ac, all divided by 2a.*

KEY CONCEPT — Quadratic Formula

The solutions of a quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the following formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

EXAMPLE Two Rational Roots

1 Solve $x^2 - 12x = 28$ by using the Quadratic Formula.

First, write the equation in the form $ax^2 + bx + c = 0$ and identify a, b, and c.

$$\begin{array}{ccc} & ax^2 + \quad bx + \quad c = 0 \\ & \downarrow \quad \downarrow \quad \downarrow \\ x^2 - 12x = 28 & \rightarrow \quad 1x^2 - 12x - 28 = 0 \end{array}$$

Then, substitute these values into the Quadratic Formula.

> **Study Tip**
>
> **Quadratic Formula**
>
> Although factoring may be an easier method to solve the equations in Examples 1 and 2, the Quadratic Formula can be used to solve any quadratic equation.

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Quadratic Formula

$= \dfrac{-(-12) \pm \sqrt{(-12)^2 - 4(1)(-28)}}{2(1)}$ Replace a with 1, b with -12, and c with -28.

$= \dfrac{12 \pm \sqrt{144 + 112}}{2}$ Simplify.

$= \dfrac{12 \pm \sqrt{256}}{2}$ Simplify.

$= \dfrac{12 \pm 16}{2}$ $\sqrt{256} = 16$

$x = \dfrac{12 + 16}{2}$ or $x = \dfrac{12 - 16}{2}$ Write as two equations.

$= 14$ $\quad = -2$ Simplify.

The solutions are -2 and 14. Check by substituting each of these values into the original equation.

CHECK Your Progress

Solve each equation by using the Quadratic Formula.

1A. $x^2 + 6x = 16$ **1B.** $2x^2 + 25x + 33 = 0$

When the value of the radicand in the Quadratic Formula is 0, the quadratic equation has exactly one rational root.

EXAMPLE One Rational Root

2 Solve $x^2 + 22x + 121 = 0$ by using the Quadratic Formula.

Identify a, b, and c. Then, substitute these values into the Quadratic Formula.

> **Study Tip**
>
> **Constants**
>
> The constants a, b, and c are not limited to being integers. They can be irrational or complex.

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Quadratic Formula

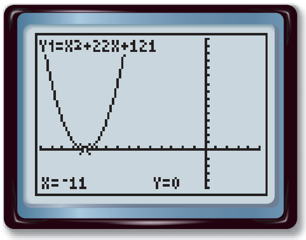
$= \dfrac{-22 \pm \sqrt{(22)^2 - 4(1)(121)}}{2(1)}$ Replace a with 1, b with 22, and c with 121.

$= \dfrac{-22 \pm \sqrt{0}}{2}$ Simplify.

$= \dfrac{-22}{2}$ or -11 $\sqrt{0} = 0$

The solution is -11.

CHECK A graph of the related function shows that there is one solution at $x = -11$.



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CHECK Your Progress

Solve each equation by using the Quadratic Formula.

2A. $x^2 - 16x + 64 = 0$ **2B.** $x^2 + 34x + 289 = 0$

You can express irrational roots exactly by writing them in radical form.

EXAMPLE 3 Irrational Roots

Solve $2x^2 + 4x - 5 = 0$ by using the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

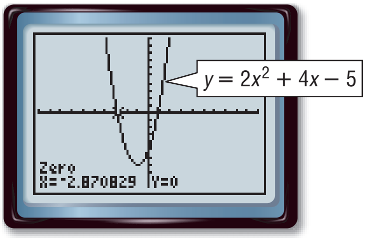
$$= \frac{-4 \pm \sqrt{(4)^2 - 4(2)(-5)}}{2(2)}$$ Replace a with 2, b with 4, and c with -5.

$$= \frac{-4 \pm \sqrt{56}}{4}$$ Simplify.

$$= \frac{-4 \pm 2\sqrt{14}}{4} \text{ or } \frac{-2 \pm \sqrt{14}}{2}$$ $\sqrt{56} = \sqrt{4 \cdot 14}$ or $2\sqrt{14}$

The approximate solutions are -2.9 and 0.9.

CHECK Check these results by graphing the related quadratic function, $y = 2x^2 + 4x - 5$. Using the ZERO function of a graphing calculator, the approximate zeros of the related function are -2.9 and 0.9.



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CHECK Your Progress

Solve each equation by using the Quadratic Formula.

3A. $3x^2 + 5x + 1 = 0$ **3B.** $x^2 - 8x + 9 = 0$

When using the Quadratic Formula, if the radical contains a negative value, the solutions will be complex. Complex solutions of quadratic equations with real coefficients always appear in conjugate pairs.

EXAMPLE 4 Complex Roots

Solve $x^2 - 4x = -13$ by using the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Quadratic Formula

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$ Replace a with 1, b with -4, and c with 13.

$$= \frac{4 \pm \sqrt{-36}}{2}$$ Simplify.

$$= \frac{4 \pm 6i}{2}$$ $\sqrt{-36} = \sqrt{36(-1)}$ or $6i$

$$= 2 \pm 3i$$ Simplify.

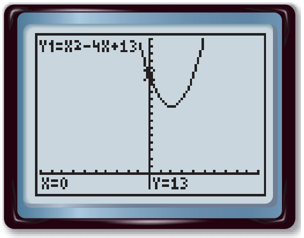
The solutions are the complex numbers $2 + 3i$ and $2 - 3i$.

Study Tip

Using the Quadratic Formula

Remember that to correctly identify a, b, and c for use in the Quadratic Formula, the equation must be written in the form $ax^2 + bx + c = 0$.

A graph of the related function shows that the solutions are complex, but it cannot help you find them.



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CHECK The check for $2 + 3i$ is shown below.

$$x^2 - 4x = -13 \quad \text{Original equation}$$
$$(2 + 3i)^2 - 4(2 + 3i) \stackrel{?}{=} -13 \quad x = 2 + 3i$$
$$4 + 12i + 9i^2 - 8 - 12i \stackrel{?}{=} -13 \quad \text{Square of a sum; Distributive Property}$$
$$-4 + 9i^2 \stackrel{?}{=} -13 \quad \text{Simplify.}$$
$$-4 - 9 = -13 \checkmark \quad i^2 = -1$$

CHECK Your Progress

Solve each equation by using the Quadratic Formula.

4A. $3x^2 + 5x + 4 = 0$ **4B.** $x^2 - 6x + 10 = 0$

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Reading Math

Roots Remember that the solutions of an equation are called *roots.*

Roots and the Discriminant

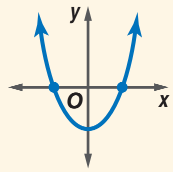
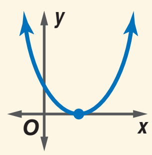
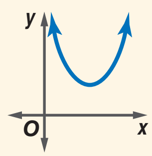
In Examples 1, 2, 3, and 4, observe the relationship between the value of the expression under the radical and the roots of the quadratic equation. The expression $b^2 - 4ac$ is called the **discriminant**.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \longleftarrow \text{discriminant}$$

The value of the discriminant can be used to determine the number and type of roots of a quadratic equation. The following table summarizes the possible types of roots.

KEY CONCEPT *Discriminant*

Consider $ax^2 + bx + c = 0$, where a, b, and c are rational numbers.

Value of Discriminant	Type and Number of Roots	Example of Graph of Related Function
$b^2 - 4ac > 0$; $b^2 - 4ac$ is a perfect square.	2 real, rational roots	
$b^2 - 4ac > 0$; $b^2 - 4ac$ is *not* a perfect square.	2 real, irrational roots	
$b^2 - 4ac = 0$	1 real, rational root	
$b^2 - 4ac < 0$	2 complex roots	

The discriminant can help you check the solutions of a quadratic equation. Your solutions must match in number and in type to those determined by the discriminant.

EXAMPLE 5 Describe Roots

Find the value of the discriminant for each quadratic equation. Then describe the number and type of roots for the equation.

a. $9x^2 - 12x + 4 = 0$

$a = 9, b = -12, c = 4$ Substitution

$b^2 - 4ac = (-12)^2 - 4(9)(4)$ Simplify.

$= 144 - 144$ Subtract.

$= 0$

The discriminant is 0, so there is one rational root.

b. $2x^2 - 16x + 33 = 0$

$a = 2, b = 16, c = 33$ Substitution

$b^2 - 4ac = (16)^2 - 4(2)(33)$ Simplify.

$= 256 - 264$ Subtract.

$= -8$

The discriminant is negative, so there are two complex roots.

CHECK Your Progress

5A. $-5x^2 + 8x - 1 = 0$

5B. $-7x + 15x^2 - 4 = 0$

You have studied a variety of methods for solving quadratic equations. The table below summarizes these methods.

CONCEPT SUMMARY *Solving Quadratic Equations*

Method	Can be Used	When to Use
Graphing	sometimes	Use only if an exact answer is not required. Best used to check the reasonableness of solutions found algebraically.
Factoring	sometimes	Use if the constant term is 0 or if the factors are easily determined. **Example** $x^2 - 3x = 0$
Square Root Property	sometimes	Use for equations in which a perfect square is equal to a constant. **Example** $(x + 13)^2 = 9$
Completing the Square	always	Useful for equations of the form $x^2 + bx + c = 0$, where b is even. **Example** $x^2 + 14x - 9 = 0$
Quadratic Formula	always	Useful when other methods fail or are too tedious. **Example** $3.4x^2 - 2.5x + 7.9 = 0$

Study Tip

Study Notebook

You may wish to copy this list of methods to your math notebook or Foldable to keep as a reference as you study.

CHECK Your Understanding

Examples 1–4 (pp. 277–279)

Find the exact solutions by using the Quadratic Formula.

1. $8x^2 + 18x - 5 = 0$
2. $x^2 + 8x = 0$
3. $4x^2 + 4x + 1 = 0$
4. $x^2 + 6x + 9 = 0$
5. $2x^2 - 4x + 1 = 0$
6. $x^2 - 2x - 2 = 0$
7. $x^2 + 3x + 8 = 5$
8. $4x^2 + 20x + 25 = -2$

Examples 3 and 4 (pp. 278–279)

PHYSICS For Exercises 9 and 10, use the following information.
The height $h(t)$ in feet of an object t seconds after it is propelled straight up from the ground with an initial velocity of 85 feet per second is modeled by the equation $h(t) = -16t^2 + 85t$.

9. When will the object be at a height of 50 feet?
10. Will the object ever reach a height of 120 feet? Explain your reasoning.

Example 5 (p. 280)

Complete parts a and b for each quadratic equation.

a. **Find the value of the discriminant.**
b. **Describe the number and type of roots. Do your answers for Exercises 1, 3, 5, and 7 fit these descriptions, respectively?**

11. $8x^2 + 18x - 5 = 0$
12. $4x^2 + 4x + 1 = 0$
13. $2x^2 - 4x + 1 = 0$
14. $x^2 + 3x + 8 = 5$

Exercises

HOMEWORK HELP

For Exercises	See Examples
15, 16	1, 5
17, 18	2, 5
19–22	3, 5
23, 24	4, 5
25–33	1–4

Complete parts a–c for each quadratic equation.

a. **Find the value of the discriminant.**
b. **Describe the number and type of roots.**
c. **Find the exact solutions by using the Quadratic Formula.**

15. $-12x^2 + 5x + 2 = 0$
16. $-3x^2 - 5x + 2 = 0$
17. $9x^2 - 6x - 4 = -5$
18. $25 + 4x^2 = -20x$
19. $x^2 + 3x - 3 = 0$
20. $x^2 - 16x + 4 = 0$
21. $x^2 + 4x + 3 = 4$
22. $2x - 5 = -x^2$
23. $x^2 - 2x + 5 = 0$
24. $x^2 - x + 6 = 0$

Solve each equation by using the method of your choice. Find exact solutions.

25. $x^2 - 30x - 64 = 0$
26. $7x^2 + 3 = 0$
27. $x^2 - 4x + 7 = 0$
28. $2x^2 + 6x - 3 = 0$
29. $4x^2 - 8 = 0$
30. $4x^2 + 81 = 36x$

FOOTBALL For Exercises 31 and 32, use the following information.
The average NFL salary $A(t)$ (in thousands of dollars) can be estimated using $A(t) = 2.3t^2 - 12.4t + 73.7$, where t is the number of years since 1975.

31. Determine a domain and range for which this function makes sense.
32. According to this model, in what year did the average salary first exceed one million dollars?
33. **HIGHWAY SAFETY** Highway safety engineers can use the formula $d = 0.05s^2 + 1.1s$ to estimate the minimum stopping distance d in feet for a vehicle traveling s miles per hour. The speed limit on Texas highways is 70 mph. If a car is able to stop after 300 feet, was the car traveling faster than the Texas speed limit? Explain your reasoning.

Complete parts a–c for each quadratic equation.

a. Find the value of the discriminant.

b. Describe the number and type of roots.

c. Find the exact solutions by using the Quadratic Formula.

34. $x^2 + 6x = 0$ **35.** $4x^2 + 7 = 9x$ **36.** $3x + 6 = -6x^2$

37. $\frac{3}{4}x^2 - \frac{1}{3}x - 1 = 0$ **38.** $0.4x^2 + x - 0.3 = 0$ **39.** $0.2x^2 + 0.1x + 0.7 = 0$

Solve each equation by using the method of your choice. Find exact solutions.

40. $-4(x + 3)^2 = 28$ **41.** $3x^2 - 10x = 7$ **42.** $x^2 + 9 = 8x$

43. $10x^2 + 3x = 0$ **44.** $2x^2 - 12x + 7 = 5$ **45.** $21 = (x - 2)^2 + 5$

Real-World Link

The Golden Gate, located in San Francisco, California, is the tallest bridge in the world, with its towers extending 746 feet above the water and the floor of the bridge extending 220 feet above the water.

Source: www.goldengatebridge.org

BRIDGES For Exercises 46 and 47, use the following information.

The supporting cables of the Golden Gate Bridge approximate the shape of a parabola. The parabola can be modeled by $y = 0.00012x^2 + 6$, where x represents the distance from the axis of symmetry and y represents the height of the cables. The related quadratic equation is $0.00012x^2 + 6 = 0$.

46. Calculate the value of the discriminant.

47. What does the discriminant tell you about the supporting cables of the Golden Gate Bridge?

48. ENGINEERING Civil engineers are designing a section of road that is going to dip below sea level. The road's curve can be modeled by the equation $y = 0.00005x^2 - 0.06x$, where x is the horizontal distance in feet between the points where the road is at sea level and y is the elevation (a positive value being above sea level and a negative being below). The engineers want to put stop signs at the locations where the elevation of the road is equal to sea level. At what horizontal distances will they place the stop signs?

H.O.T. Problems

EXTRA PRACTICE

See pages 901, 930.

Math Online

Self-Check Quiz at algebra2.com

49. OPEN ENDED Graph a quadratic equation that has a

a. positive discriminant. **b.** negative discriminant. **c.** zero discriminant.

50. REASONING Explain why the roots of a quadratic equation are complex if the value of the discriminant is less than 0.

51. CHALLENGE Find the exact solutions of $2ix^2 - 3ix - 5i = 0$ by using the Quadratic Formula.

52. REASONING Given the equation $x^2 + 3x - 4 = 0$,

a. Find the exact solutions by using the Quadratic Formula.

b. Graph $f(x) = x^2 + 3x - 4$.

c. Explain how solving with the Quadratic Formula can help graph a quadratic function.

53. *Writing in Math* Use the information on page 276 to explain how a diver's height above the pool is related to time. Explain how you could determine how long it will take the diver to hit the water after jumping from the platform.

STANDARDIZED TEST PRACTICE

54. ACT/SAT If $2x^2 - 5x - 9 = 0$, then x could be approximately equal to which of the following?

A -1.12

B 1.54

C 2.63

D 3.71

55. REVIEW What are the x-intercepts of the graph of $y = -2x^2 - 5x + 12$?

F $-\frac{3}{2}, 4$

G $-4, \frac{3}{2}$

H $-2, \frac{1}{2}$

J $-\frac{1}{2}, 2$

Spiral Review

Solve each equation by using the Square Root Property. (Lesson 5-5)

56. $x^2 + 18x + 81 = 25$

57. $x^2 - 8x + 16 = 7$

58. $4x^2 - 4x + 1 = 8$

Simplify. (Lesson 5-4)

59. $\frac{2i}{3 + i}$

60. $\frac{4}{5 - i}$

61. $\frac{1 + i}{3 - 2i}$

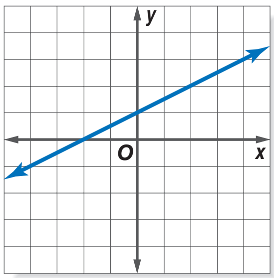
Solve each system of inequalities. (Lesson 3-3)

62. $x + y \leq 9$
$x - y \leq 3$
$y - x \geq 4$

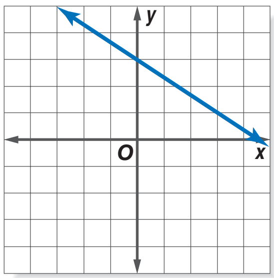
63. $x \geq 1$
$y \leq -1$
$y \leq x$

Write the slope-intercept form of the equation of the line with each graph shown. (Lesson 2-4)

64.



65.



66. PHOTOGRAPHY Desiree works in a photography studio and makes a commission of \$8 per photo package she sells. On Tuesday, she sold 3 more packages than she sold on Monday. For the two days, Victoria earned \$264. How many photo packages did she sell on these two days? (Lesson 1-3)

GET READY for the Next Lesson

PREREQUISITE SKILL State whether each trinomial is a perfect square. If so, factor it. (Lesson 5-3.)

67. $x^2 - 5x - 10$

68. $x^2 - 14x + 49$

69. $4x^2 + 12x + 9$

70. $25x^2 + 20x + 4$

71. $9x^2 - 12x + 16$

72. $36x^2 - 60x + 25$